Mathematics

Preparation for A’ Level

Complete as many questions as you can in each section. Answers are at the back but we will ask to see your completed ‘Test Yourself’ sections (with working shown) in September. The more questions you complete, the better prepared you will be for this course.

There is an optional extra folder of questions for those that are aiming for a top grade at A’ level.

As there are seven separate sections, aim to complete one each week between now and starting year 12.

Surds and indices

1 The factors of 18 are 1, 2, 3, 6, 9, 18.

The first few square numbers are 1, 4, 9, 16, 25, 36, 49, ...

The highest square factor of 18 is 9 because 9 is a square number and also a factor of 18.

Find the highest square factor of:

a 50	**b** 20	**c** 72
d 8	**e** 60	**f** 128

2 Copy and complete:

$\sqrt{9 \times 4} = \sqrt{36} =$

$\sqrt{9} \times \sqrt{4} = 3 \times \ldots = \ldots$

Your answer should show that $\sqrt{9 \times 4} = \sqrt{9} \times \sqrt{4}$

In a similar way show that:

a $\sqrt{25 \times 4} = \sqrt{25} \times \sqrt{4}$ **b** $\sqrt{9 \times 16} = \sqrt{9} \times \sqrt{16}$

c $\sqrt{49 \times 4} = \sqrt{49} \times \sqrt{4}$ **d** $\sqrt{25 \times 16} = \sqrt{25} \times \sqrt{16}$

e $\sqrt{4 \times 9 \times 16} = \sqrt{4} \times \sqrt{9} \times \sqrt{16}$

3 Use a calculator to show that:

a **i** $\sqrt{72} = \sqrt{36} \times \sqrt{2}$ **ii** $\sqrt{72} = \sqrt{24} \times \sqrt{3}$

b **i** $\sqrt{502} = \sqrt{25} \times \sqrt{2}$ **ii** $\sqrt{50} = \sqrt{10} \times \sqrt{5}$

c **i** $\sqrt{48} = \sqrt{16} \times \sqrt{3}$ **ii** $\sqrt{48} = \sqrt{24} \times \sqrt{2}$

d **i** $\sqrt{90} = \sqrt{9} \times \sqrt{10}$ **ii** $\sqrt{72} = \sqrt{30} \times \sqrt{3}$

Which of each pair of expressions in **a**, **b**, **c** and **d** involves a square number?

4 Find the value of n in each of the following:

a $2^n = 32$ **b** $5^n = 125$

c $10^n = 1$ **d** $2^n = 0.5$

e $2^n = 0.25$ **f** $10^n = 10000$

g $2^n = 8^2$ **h** $5^n = 25^2$

i $5^n = 1$ **j** $4^n = 2$

k $8^n = 2$ **l** $100^n = 10$

5 Use the x^y button on your calculator to show that:

a $3^4 \times 3^2 = 3^6$ **b** $6^5 \div 6^2 = 6^3$

c $10^{10} \div 10^{10} = 10^0$ **d** $2^3 \times 3^2 \times 2^5 \times 3^4 = 2^8 \times 3^6$

e $5^4 \div 5^5 = 5^{-1}$ **f** $16^{0.5} = 4$

g $(5^2)^3 = 5^6$ **h** $(2^5)^2 = 2^{10}$

i $25^{0.5} = 5$ **j** $100^{0.5} = 10$

k $25^{0.5} = 0.2$ **l** $100^{0.5} = 0.1$

Rational numbers and their decimal representation

All rational numbers can be expressed as either

- decimals which terminate, such as 0.72, or
- recurring decimals, such as 0.102 432 432 43 . . .

0.102 432 43 . . . can be written as $0.10\dot{2}4\dot{3}$. The dots indicate the set of digits which recur.

Conversely, all decimals that terminate or recur represent rational numbers. A method for converting a terminating decimal into a fraction is shown in the Example below.

Irrational numbers cannot be expressed as terminating or recurring decimals. Conversely an infinite decimal that never recurs cannot be expressed as a rational number. For example, in 0.101 001 0001 . . . the number of zeros increases by one before each digit 1, so the decimal never recurs and it cannot be expressed as a ratio of two integers.

To sum up: Rational numbers can be represented by terminating or recurring decimals and *vice versa*. Irrational numbers are non-terminating, non-recurring decimals and *vice versa*.

Example

Express decimals as fractions in their lowest terms.

a $43.72 = \dfrac{4372}{100}$ — 43.72 is 4372 hundredths.

10	1	$\frac{1}{10}$	$\frac{1}{100}$
4	3	7	2

Divide numerator and denominator by 4.

$= \dfrac{1093}{25}$ — The fraction cannot be cancelled further. It is in its lowest terms.

b $0.\dot{7}$

Let $x = 0.77\dot{7}$ ①

Then $10x = 7.77\dot{7}$ ② — Multiply both sides by 10. Since one digit recurs, this lines up the recurring digits.

$9x = 7$ — Subtract line ① from line ②.

$x = \dfrac{7}{9}$ — Divide both sides by 9.

c $0.\dot{2}\dot{4}$

Let $x = 0.\dot{2}\dot{4}$ ①

Then $100x = 24.\dot{2}\dot{4}$ ② — Multiply both sides by 100. Since two digits recur, this lines up the recurring digits.

$99x = 24$ — Subtract line ① from line ②.

$x = \dfrac{24}{99}$ — Divide both sides by 99.

$x = \dfrac{8}{33}$ — Cancel to lowest terms.

Extension

To prove that $\sqrt{2}$ is irrational

The ancient Greeks, who were among the most accomplished mathematicians of antiquity, believed that any number could be written as a fraction. They were troubled by the number representing the length of the diagonal of a unit square. Using Pythagoras' theorem, they knew that this number was $\sqrt{2}$. After trying in vain to find a fraction that, when squared, gave 2, the Greeks finally realised – and managed to prove – that it was impossible. Legend has it that this breakthrough was celebrated by sacrificing 100 oxen. The proof is given below.

Proving any proposition requires an argument, a series of steps, which will convince a reader of the truth of the proposition.

There are various methods of proof. The proof given here uses a method called *reductio ad absurdum* (reduction to the absurd). In this method of proof, the proposition to be proved true is assumed to be false. Logical steps are then deduced leading to a contradiction. The assumption that the proposition was false has led logically to a contradiction. Therefore that assumption must have been incorrect and the proposition must be true.

Proof

Assume that the proposition is not true, i.e. that $\sqrt{2}$ is rational.

Then $\sqrt{2}$ can be expressed as $\dfrac{p}{q}$ where p and q are integers, $(q \neq 0)$, with $\dfrac{p}{q}$ in its lowest terms.

A fraction can always be reduced to its lowest terms.

Let $\qquad \dfrac{p}{q} = \sqrt{2}$

Squaring $\qquad \dfrac{p^2}{q^2} = 2$

So $\qquad p^2 = 2q^2 \qquad (1)$

and hence p^2 is even.

Since p^2 is even, p must also be even, so one can write

$$p = 2m \quad \text{where } m \text{ is an integer}$$

Even × Even = Even
Odd × Odd = Odd

Substituting in (1) $\qquad 4m^2 = 2q^2$

Dividing both sides by 2 $\qquad 2m^2 = q^2$

So q^2 is even and therefore q is even. But if both p and q are even then $\frac{p}{q}$ is not in its lowest terms.

Since there is a contradiction, the original assumption must be false. That is, $\sqrt{2}$ cannot be rational.

$\therefore \quad \sqrt{2}$ is irrational.

It is extraordinary that $\sqrt{2}$, with just two symbols, expresses a number that can otherwise only be expressed in decimal form and, however many decimal places are given, will only be an approximation!

Extension Exercise

Calculators should not be used for this exercise.

1 Express these in the form $a\sqrt{b}$, where b has no perfect square factors.

a $\sqrt{192}$ **b** $\sqrt{1452}$ **c** $\sqrt{77\,760}$

2 Express these in the form $A + B\sqrt{C}$, where A, B and C are rational.

a $\dfrac{1}{\sqrt{6}} + \dfrac{1}{\sqrt{216}}$ **b** $\dfrac{\sqrt{8}+3}{\sqrt{18}+2}$ **c** $\sqrt{3} + 2 + \dfrac{1}{\sqrt{3}-2}$

3 Rationalise the denominators of these fractions.

a $\dfrac{2\sqrt{5}+1}{2\sqrt{5}-1}$ **b** $\dfrac{3+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ **c** $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$

d $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$ **e** $\dfrac{1}{a\sqrt{b}+\sqrt{c}}$ **f** $\dfrac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}$

4 Prove that $\sqrt{3}$ is irrational (see *To prove* $\sqrt{2}$ *is irrational* on the CD-ROM in Surds and Indices).

5 Find the value of

a $\dfrac{16^{\frac{1}{3}} \times 4^{\frac{1}{3}}}{8}$ **b** $\dfrac{27^{\frac{1}{2}} \times 243^{\frac{1}{2}}}{243^{\frac{4}{5}}}$ **c** $\dfrac{12^{\frac{1}{3}} \times 6^{\frac{1}{3}}}{81^{\frac{1}{6}}}$ **d** $\dfrac{32^{\frac{3}{4}} \times 16^{0} \times 8^{\frac{5}{4}}}{128^{\frac{3}{2}}}$

6 Simplify

a $\dfrac{8^{n} \times 2^{2n}}{4^{3n}}$ **b** $\dfrac{3^{n+1} \times 9^{n}}{27^{2n/3}}$ **c** $\dfrac{x^{-\frac{2}{3}} \times x^{\frac{1}{4}}}{x^{\frac{1}{6}}}$ **d** $\dfrac{x^{2n+1} \times x^{\frac{1}{2}}}{\sqrt{x^{3n}}}$

7 Prove that $3^{\frac{1}{3}} > 2^{\frac{1}{2}}$.

8 Insert brackets to make this a true statement: $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}} = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$

9 Given $3^{x} = 2$ and $3^{y} = 125$, calculate

a 3^{x+y} **b** 3^{2x} **c** 3^{y+1} **d** 3^{-x}

e 3^{-y} as a decimal **f** 3^{y-x} **g** $3^{y/3}$ **h** 9^{x+1}

10 Find these roots by expressing each number in prime factors.

a $\sqrt{9801}$ **b** $\sqrt[3]{2744}$ **c** $\sqrt[5]{7776}$ **d** $\sqrt[4]{20\,736}$

e $\sqrt{7744}$ **f** $\sqrt[3]{421875}$

11 Simplify

a $\left(\left(x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$ **b** $\left(\left(x^{\frac{1}{3}}\right)^{\frac{2}{5}}\right)^{-30}$

12 If $x^{y} = y^{x}$ prove that $\left(\dfrac{x}{y}\right)^{\frac{x}{y}} = x^{\frac{x}{y}-1}$.

13 Simplify these, where possible.

a $a^4 - a^3$	**b** $3b^6 - 3b^3$	**c** $7c^5 - 7c^3$
d $\frac{3d^6}{3d^2}$	**e** $\frac{10e^3}{2e^2}$	**f** $\frac{49f^4}{7f^2}$
g $\sqrt{g^6}$	**h** $\sqrt[3]{h^{12}}$	**i** $\sqrt{25i^4}$
j $9j^5 - 8j^5$	**k** $3k^3 + 4k^4$	**l** $\sqrt[4]{l^8m^{20}}$
m $\sqrt{4n^2p^2}$	**n** $\sqrt{q^4}$	**o** $r^7 + r^7$
p $3s^4 + 4s^4$	**q** $15t^9 - 15t^9$	**r** $u^2 - u$

Test yourself

1 $(3\sqrt{5})^2$ is equal to

A 45 **B** 15 **C** $9\sqrt{5}$ **D** $\sqrt{\sqrt{45}}$ **E** $6\sqrt{5}$

2 $(a^6)^{-2}$ is equivalent to

A a^4 **B** $\frac{1}{a^{12}}$ **C** a^3 **D** $\frac{1}{a^3}$ **E** $-\sqrt{a^6}$

3 The expression $4\sqrt{63} - 5\sqrt{28}$ is equal to

A $-\sqrt{35}$ **B** $2\sqrt{7}$ **C** $16\sqrt{7}$

D $\sqrt{308}$ **E** none of these

4 $7 \times 10^{100} + 8 \times 10^{102}$ is equal to

A 1.5×10^{102} **B** 5.6×10^{101} **C** 7.08×10^{100}

D 8.07×10^{102} **E** 1.5×10^{203}

5 $\frac{(2a^2b)^3}{(ab)^5}$ is equivalent to

A $\frac{6}{a^3b^4}$ **B** $\frac{8a}{b^2}$ **C** $\frac{8}{b^2}$ **D** $(2a)^{\frac{3}{5}}$ **E** $8a^{-9}b^{-12}$

6 $\sqrt{12}\sqrt{15}\sqrt{20}$ is equal to

A 60 **B** $60\sqrt{15}$ **C** $30\sqrt{2}$ **D** $60\sqrt{12}$ **E** $\sqrt{\sqrt{60}}$

7 $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ is equal to

A $\frac{64}{19\,683}$ **B** $\frac{64}{729}$ **C** $21\frac{1}{3}$ **D** $\frac{4}{9}$ **E** $\frac{2\sqrt{2}}{3}$

8 When two surds are multiplied, the result is

A always a surd

B never a perfect square

C never rational

D sometimes rational

E either irrational or prime

9 $\frac{\sqrt{6}}{2+\sqrt{3}}$ is equal to

A $\sqrt{2} + \frac{\sqrt{6}}{2}$ **B** $2\sqrt{6} - 3\sqrt{2}$ **C** $\frac{12+3\sqrt{2}}{7}$ **D** $3\sqrt{2} - 2\sqrt{6}$ **E** $\sqrt{12} - \sqrt{6}$

10 $\left(\frac{a^4b}{c^2}\right)^{-\frac{1}{2}}$ is equivalent to

A $-\frac{a^2\sqrt{b}}{c}$ **B** $\frac{a^{\frac{7}{2}}b^{-\frac{1}{2}}}{c^{\frac{3}{2}}}$ **C** $\frac{c}{a^4b}$ **D** $\frac{c\sqrt{b}}{a^2b}$

E none of these

Algebraic expressions

1 **a** When $x = -2$ find the value of:

i x^2 **ii** $2x$ **iii** $x^2 - 2x$ **iv** $x(x-2)$

b When $y = 3$ find the value of:

i y^3 **ii** $3y$ **iii** $y^3 - 3y$ **iv** $y(y^2 - 3)$

c When $y = 3$ and $y = -2$ find the value of:

i $2x + 4y$ **ii** $2(x + 2y)$ **iii** $3x - 9y$ **iv** $3(x - 3y)$

d When $p = 5$ and $q = 4$ find the value of:

i $p^2 - q^2$ **ii** $(p - q)(p + q)$

2 Work out the value of $x^3 + 2x^2 - 3x + 16$ when:

a $x = 0$ **b** $x = 2$ **c** $x = -1$ **d** $x = \frac{1}{2}$

3 **a** Simplify the following by collecting like terms:

i $3x - 2y - 5x + 7y$ **ii** $5a + 7b - 8a - 3b + a$

iii $2x^2 - 3xy - y^2 + 2xy + 4y^2$ **iv** $-3a^2 - 2b^2 + a^2 + 3ab - 3b^2$

b Expand and simplify the following:

i $2(x + 4) + 3(2x - 1)$ **ii** $2x(3x + 4) - x(2x - 1)$

iii $(x + 2)(x + 4)$ **iv** $(x - 4)(x + 3)$

4 Write down the highest common factor (HCF) of:

a 12 and 30 **b** $12x$ and $30y$ **c** $12x$ and $30x$

d $12x$ and $30x^2$ **e** $36x$ and $54y$ **f** $36x$ and $54x^2$

g $36xy$ and $54y^2$ **h** $10a$ and $15a$ **i** $10a^2$ and $15a$

j $10a^2$ and $15a^3$ **k** ab and a^2 **l** x^2y and xy^2

5 Factorise the following:

a $2a + 4$ **b** $6x + 10$ **c** $x^2 + x$

d $ab + 2a^2$ **e** $10x^2 + 15x$ **f** $6a^2 - 9a^3$

Factors, including highest common factors

A **factor** divides exactly into an integer or algebraic expression.

When a factor divides into two or more integers or algebraic expressions it is called a **common factor**.

So 5 is a common factor of 10, 15, 20
a^2b is a common factor of a^2b^2, a^3bc

The **highest common factor** (**HCF**) of two (or more) integers or algebraic expressions is the largest factor common to both (or all).

The common factors of 12, 42 and 54 are 1, 2, 3 and 6.

So the HCF is 6.

6 is the largest of all the common factors.

The common factors of a^2bc and a^2b^3 are 1, a, ab, a^2b.

So the HCF is a^2b.

The HCF is the expression with the highest powers of a and b.

Example 1 Find the HCF of these numbers.

a 14 and 24

$14 = 2 \times 7$

$24 = 2 \times 12$

The only common factor, other than 1, is 2.
This can be seen by inspection.
7 and 12 share no common factor.

The HCF of 14 and 24 is 2.

b 9 and 25

The HCF of 9 and 25 is 1, because 9 and 25 have no common factor other than 1.

Note Any numbers whose HCF is 1 are called **co-prime**.

c 56, 168 and 140

To find the HCF of 56, 168 and 140, first express each integer as the product of prime numbers.

$56 = 8 \times 7$

$= 2^3 \times 7$

$168 = 7 \times 24$

$= 7 \times 8 \times 3$

$= 2^3 \times 3 \times 7$

$56 = 2^3 \times 7$
$168 = 2^3 \times 3 \times 7$
$140 = 2^2 \times 5 \times 7$

$140 = 10 \times 14$

$= 2^2 \times 5 \times 7$

$\text{HCF} = 2^2 \times 7 = 28$

The highest power of 2 which appears in every number is 2^2.
The other factor which appears in every number is 7.
So $2^2 \times 7$ is the HCF (the product of the highest powers).

The HCF of 56, 168 and 140 is 28.

Note Expressing integers as the product of prime factors is helped by using divisibility tests (see the Glossary).

Example 2 Find the HCF of these algebraic expressions.

a $21a^4x^2$, $35a^2x^4$ and $28a^3x$

The HCF of 21, 35 and 28 is 7.

First look at the coefficients.

The highest power of a which divides into a^4, a^2 and a^3 is a^2.

Then a.

The highest power of x which divides into x^2, x^4 and x is x.

Then x.

So HCF is $7a^2x$.

b $12(x^2 - 4)$, $6x - 12$ and $18x - 24 - 3x^2$

The expressions must first be factorised.

$$12(x^2 - 4) = 12(x - 2)(x + 2)$$

$$6x - 12 = 6(x - 2)$$

$$\begin{aligned} 18x - 24 - 3x^2 &= 3(6x - 8 - x^2) \\ &= 3(2 - x)(x - 4) \\ &= 3(x - 2)(4 - x) \end{aligned}$$

Multiplying both brackets by -1 does not alter the value of the expression.

So the HCF is $3(x - 2)$.

The HCF of the coefficients 12, 6 and 3 is 3. $(x - 2)$ is a factor of each expression.

Multiples, including lowest common multiples

An integer or algebraic expression divides into its **multiples** exactly.

A **common multiple** of two (or more) numbers or algebraic expressions is one into which both (or all) divide exactly. So

- common multiples of 4 and 6 are 12, 24, 36, . . .
- common multiples of a^2 and ab^2 include a^2b^2, a^3b^2, a^3b^3, $6a^2b^2$.

The **lowest (or least) common multiple (LCM)** of two (or more) integers or algebraic expressions is the lowest multiple common to both (or all).

- The LCM of 4 and 6 is 12, since 12 is the lowest of the common multiples.
- The LCM of a^2 and ab^2 is a^2b^2.

This is the expression with the lowest powers of a and b into which a^2 and, separately, ab^2 can be divided.

Example 3 Find the LCM of the numbers 21, 35 and 28.

$21 = 3 \times 7$ — To be a multiple of 21, the LCM must contain the factors 3 and 7

$35 = 5 \times 7$ — To be a multiple of 35, the LCM must contain 5 and 7

$28 = 2^2 \times 7$ — To be a multiple of 28, the LCM must contain 2^2 and 7

$\text{LCM} = 2^2 \times 3 \times 5 \times 7$

$= 420$ — The least set of factors required for a multiple of 21, 35 and 28 is $2^2 \times 3 \times 5 \times 7$.

Example 4 Find the LCM of the algebraic expressions $21a^4x^2y$, $35a^2x^4y$ and $28a^3x^3y^3$.

The LCM of 21, 35, 28 is 420 (see Example 9).

The lowest power of a divisible by a^4, a^2 and a^3 is a^4.

The lowest power of x divisible by x^2, x^4 and x^3 is x^4.

The lowest power of y divisible by y, y and y^3 is y^3.

So the LCM is $420a^4x^4y^3$.

Some uses of HCFs and LCMs

- HCFs are used when cancelling fractions, and in factorising.
- LCMs are used when adding and subtracting fractions.

Exercise

1 Find the HCF and LCM of

a 12, 15 **b** 7, 8 **c** 30, 36, 40

d 10, 12, 14 **e** 21, 30, 35 **f** 9, 14, 25

2 Find the HCF and LCM of

a $a(a+b)$, $b(a+b)$ **b** xy, x^3y^2

c $2x^3$, $4x^2y$ **d** $3ab^3$, $2a^2b$

e cd^5, $4c^3d$ **f** $2cd$, $4c^2$, $6abc$

g $12ab^2c$, $8a^2b^4$, $4a^3bc^2$ **h** $51p^4q^2r^3$, $34p^2r^5$, $17p^3r^4$

i $(3x+1)(2x-1)$, $x^2(3x+1)$ **j** $2x(x+1)(x-1)$, $x^2(x-1)$

k $ab^2(b+c)$, $a^2b(b+c)^2$ **l** $4(a+b)(c+d)$, $12(c+d)$

3 Find the HCF and LCM.

a $2a, \quad 3a$

b $ac, \quad bc$

c $2x^2y, \quad 5xy^2$

d $2a+6, \quad 3a+9$

e $3b-3, \quad 4b-4$

f $4a-8d, \quad 6a-12d$

g $a^2-ab, \quad b^2-ab$

h $2abc, \quad 6a^2b, \quad 10bc^2$

i $6ab, \quad 5cd$

j $x^2-3x-10, \quad x^2-6x+5$

k $x^2-5x+6, \quad x^2-x-6$

l $2a-6b, \quad a^2-9b^2$

m $2pq^2, \quad 8p^4q$

n $2x^3, \quad 4x^2y$

o $2cd, \quad 4c^2, \quad 6abc$

4 Find the HCF and LCM.

a $c^2-c, \quad (c-1)^2, \quad c^3-1$

b $d+2, \quad d^2-4, \quad d^3+8$

c $15a^2+8a+1, \quad 12a^2+a-1$

d $a^2-ab-2b^2, \quad a^2+3ab+2b^2$

e $x^4-27x, \quad x^4+2x^3-15x^2$

f $x^3-x, \quad 2x^4-3x^3+x^2$

Exercise

This exercise revises numerical fractions. Do not use a calculator.

1 Fill in the missing terms.

a $\frac{2}{5} = \frac{\square}{10}$ **b** $\frac{3}{7} = \frac{15}{\square}$ **c** $\frac{2}{9} = \frac{\square}{810}$

d $\frac{3}{2} = \frac{\square}{8}$ **e** $2 = \frac{\square}{8}$

2 Find the value of

a $\frac{6}{7} + \frac{3}{14}$ **b** $\frac{1}{5} - \frac{1}{4} + \frac{1}{10}$ **c** $3\frac{1}{4} + 2\frac{4}{9}$

d $3\frac{1}{4} - 2\frac{4}{9}$ **e** $\frac{4}{5} + \frac{2}{5} + \frac{7}{10}$ **f** $3\frac{1}{2} + 11\frac{2}{5}$

g $7\frac{2}{7} - 5\frac{2}{5}$ **h** $107\frac{1}{4} - 41\frac{2}{3}$ **i** $1001\frac{1}{3} + 98\frac{1}{4}$

3 Simplify

a $12 \times \frac{3}{4}$ **b** $\frac{2}{3} \times \frac{9}{8}$ **c** $4\frac{1}{4} \times \frac{8}{51}$

d $2\frac{1}{3} \times 4\frac{1}{5}$ **e** $5\frac{2}{3} \times 3$ **f** $5\frac{2}{3} \times 4\frac{1}{2}$

g $120 \times \frac{3}{5}$

4 Simplify

a $1\frac{1}{2} \times 1\frac{1}{3} \times 1\frac{1}{4} \times 1\frac{1}{5} \times 1\frac{1}{6}$ **b** $1\frac{2}{3} \times 1\frac{2}{5} \times 1\frac{2}{7} \times 1\frac{2}{9} \times 1\frac{2}{11} \times 1\frac{2}{13}$

5 Simplify

a $2\frac{2}{5} \div 1\frac{1}{3}$ **b** $\frac{3}{10} \div 2\frac{2}{5}$ **c** $\frac{3}{4} \div \frac{5}{12}$

d $1\frac{2}{3} \div 2$ **e** $2\frac{7}{10} \div 7\frac{1}{5}$ **f** $4\frac{1}{3} \div 3\frac{1}{4}$

Extension Exercise

1 Given $x = 129$ and $y = 71$, use factors to find the value of

a $\dfrac{x^2 - y^2}{x - y}$ b $\dfrac{x + y}{x^2 - y^2}$ c $x^2 + 2xy + y^2$

2 Factorise $xy + x + y + 1$.
Hence show that there are no positive integer solutions to $xy + x + y = 30$.

3 Find the HCF of $1144x^n$ and $585x^{2n}$.

4 Simplify

a $x - 1 + \dfrac{1}{1 + x}$ b $\dfrac{N(4N^2 - 1) + 3(2N + 1)^2}{N + 1}$

c $\dfrac{(x + h)^3 - x^3}{h}$ d $\dfrac{\frac{a}{b} + \frac{c}{d}}{1 + \frac{ac}{bd}}$ e $1 - \dfrac{1}{1 + \frac{1}{a}}$

f $\dfrac{\frac{2}{x^2} + \frac{3}{x}}{\frac{5}{x^2} - y}$ g $\dfrac{2 - \frac{4}{x} - \frac{6}{x^2}}{1 - \frac{3}{x} - \frac{4}{x^2}}$ h $\dfrac{5x - \frac{20}{x}}{3x - 3 - \frac{6}{x}}$

i $\dfrac{1 - a^2}{a - 1}$ j $\sqrt{(a - b)^2 + 4ab}$ k $\dfrac{2T - 2t}{T^2 - t^2}$

l $\dfrac{1 - \frac{1}{t}}{1 - t}$ m $\dfrac{T - t}{\frac{1}{T} - \frac{1}{t}}$ n $(x^2 + 1)^{\frac{1}{2}} - \dfrac{1}{(x^2 + 1)^{\frac{1}{2}}}$

o $(1 + x)^{\frac{1}{3}} - x(1 + x)^{-\frac{2}{3}}$

5 Express each of these as a single fraction.

a $\dfrac{1}{(x - h)^2} - \dfrac{1}{(x + h)^2}$ b $\dfrac{2}{(N + 1)(N + 3)} - \dfrac{2N + 3}{(N + 1)(N + 2)}$

c $\dfrac{2n + 1}{n^2(n + 1)^2} - \dfrac{1}{n^2}$ d $\dfrac{1}{a + 1} + \dfrac{2}{a - 1} + \dfrac{3}{a^2 - 1}$

e $\dfrac{2}{x - 1} - \dfrac{3}{x + 1}$ f $\dfrac{7}{x} + \dfrac{2}{x + 1} + 1$

g $\dfrac{1}{a + b} + \dfrac{1}{a - b}$ h $\dfrac{3}{x - 2} - \dfrac{2}{x + 2}$

i $\dfrac{u - v}{6} - \dfrac{u + v}{8}$ j $7 - \dfrac{1}{7 - x}$

k $\dfrac{1}{x + 3} + \dfrac{3}{(x^2 - 9)}$ l $\dfrac{6}{x^2 - 2x - 8} + \dfrac{1}{x^2 + 5x + 6}$

m $\dfrac{7}{x^2 + x - 12} - \dfrac{6}{x^2 + 2x - 8}$ n $\dfrac{p + 2}{2} - \dfrac{p}{p + 2} - \dfrac{p^3 - 2p^2}{2p^2 - 8}$

o $\dfrac{1}{3a - 1} + \dfrac{2}{a - 1} + \dfrac{1}{a}$

6 Simplify

a $\dfrac{m+n}{m}\times\dfrac{mn}{3m+3n}$ **b** $\dfrac{a^2-b^2}{ab+a^2}\times\dfrac{2a^2}{ab-a^2}$

c $\dfrac{3d^2-12}{9d^2}\times\dfrac{6d^3}{4d+8}$ **d** $\dfrac{cd}{c^2+cd}\times\dfrac{c^2-d^2}{d^4}$

e $\dfrac{13c^2}{15}\times\dfrac{16a^4}{39d^3}\times\dfrac{27d^4}{48a^2b^2c}$ **f** $\dfrac{a^2-b^2}{x^2-y^2}\times\dfrac{x+y}{a-b}$

g $\dfrac{a^2-4}{a^2-3a+2}\div\dfrac{a}{a-1}$ **h** $\dfrac{a^2-b^2}{a^2+ab}\div\dfrac{2a-2b}{ab}$

i $\dfrac{u^2+3u-10}{3u^2+12u}\div\dfrac{u^2-25}{u^2-u-20}$ **j** $\dfrac{4x^2}{x^2-3x}\times\dfrac{2(x-3)}{x^3}$

k $\dfrac{3+x}{(2-x)(4+x)}\times\dfrac{(x+4)}{6(x+3)}$ **l** $\dfrac{x^2-1}{x^2-3x+2}\times\dfrac{x^2-4}{2(x+1)}$

m $\dfrac{2x+2}{x^2-x-6}\times\dfrac{4x^2+8x-60}{x+1}$ **n** $\dfrac{x^2-4x+3}{x^2+4x+3}\times\dfrac{x^2+6x+9}{x^2-9}$

o $\dfrac{3x^2+x}{x+4}\times\dfrac{x+4}{3x^2-2x-1}$ **p** $\dfrac{a^2-b^2}{a^2+2ab+b^2}\div\dfrac{4(a^2-ab)}{a^2+ab}$

q $\dfrac{x^2+3x+2}{x+3}\div\dfrac{x^2+4x+3}{x+2}$ **r** $\dfrac{x^2+7x+10}{x^2+3x-18}\div\dfrac{x^2+2x-15}{x^2+8x+12}$

s $\dfrac{x^3-xy^2}{(x+y)^2}\div\dfrac{(x-y)^2}{x^2y-y^3}$ **t** $\dfrac{2-x-x^2}{8-2x-3x^2}\div\dfrac{x^2-x}{3x-4}$

7 Given that $f(x)=3x^2-x+2$, and $g(x)=7x^4-2x^3+x^2+1$, find

a $f(x)+g(x)$ **b** $f(x)-2g(x)$ **c** $f(x)g(x)$ **d** $x^3f(x)+xg(x)$

8 Find the coefficient of the x^3 term in these expansions.

a $(x^4+3x^2-2x+1)(x^3-2x^2+x-7)$

b $(2x^3-x^2+5x-4)(-x^3+4x^2+2x+1)$

9 Expand and simplify

a $(7x^2+5x-1)(6x^3-3x^2+2x+4)$ **b** $(x^2+2xy+y^2)(x^2-2xy+y^2)$

c $(y^3+3y^2-2y-2)(4y^4-2y+7)$ **d** $(u^3+v^2-2w)^2$

e $(3xy+1)(2x^2-3y)$ **f** $(2x+1)(3x-1)(2x+7)$

Test yourself

1 The sum of $a^3 + a^2b - ab^2 + 2$ and $2a^3 + 2a^2b - 4$ is

A $2a^6 + 2a^4b^2 - ab^2 - 2$ B $2a^6 + 4a^5b - 2a^3b^3 + 4ab^2 - 8$

C $3a^3 + 3a^2b - ab^2 - 2$ D $2a^9 - 8$

E $3a^6 + 3a^4b^2 - ab^2 - 8$

2 The complete factorisation of $2p^3 - 2p^2 - 4p$ is

A $2(p^3 - p^2 - 2p)$ B $2p(p+1)(p-2)$ C $0, -1$ or 2

D $4p^3\left(\frac{1}{2} - \frac{1}{2p} - \frac{1}{p^2}\right)$ E $2p^2(-3p - 1)$

3 $(3r+1)(r-2) - (r-2)(r+1)$ is equivalent to

A $2r^2 - 6r - 4$ B $-3r^4 + 8r^3 + 3r^2 - 12r - 4$ C $2r^2 - 4r$

D $2(r+1)(r-2)$ E 0

Equations and quadratic functions

1 Solve each of the following equations:

a $3x = 2$
b $x - 3 = -7$
c $5x - 1 = 3$
d $3x - 2 = 7x + 2$
e $4x + 3 = -1$
f $8x - 2 = 6x - 1$
g $5x - 7 = 2x - 1$
h $7 - 3x = 5x - 9$

2 Make x the subject of each of these formulae.

a $x + a = b$
b $x - a = b$
c $ax = b$
d $ax - b = c$
e $\frac{x}{a} = b$
f $\sqrt{x} = a$
g $a\sqrt{x} = b$
h $x^2 + a = b$

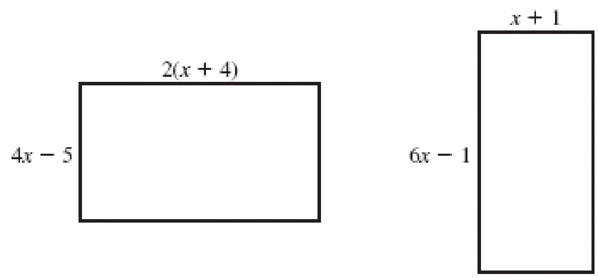
3 Factorise each of these quadratic expressions.

a $x^2 + 3x + 2$
b $x^2 + 5x + 6$
c $x^2 + 7x + 12$
d $x^2 - x - 2$
e $x^2 + x - 2$
f $x^2 + x - 6$
g $x^2 - 9x + 8$
h $x^2 - 7x + 10$

4 To solve a quadratic equation it needs to be written in the form $ax^2 + bx + c = 0$. Sometimes this means that the equation needs to be rearranged.
For example: $3x^2 + 2x - 3 = x^2 - 3x + 7$
Step 1 Take x^2 from both sides to give
$2x^2 + 2x - 3 = -3x + 7$
Step 2 Add $3x$ to both sides to give
$2x^2 + 5x - 3 = 7$
Step 3 Take 7 from both sides to give
$2x^2 + 5x - 10 = 0$
In this quadratic equation $a = 2$, $b = 5$ and $c = -10$.
Write each of these quadratic equations in the form $ax^2 + bx + c$ and then write down the values of a, b and c.

a $5x^2 + 3x - 1 = 2x^2 - x + 4$
b $3x^2 - 2x - 3 = x^2 + 3x - 1$
c $13x^2 + x - 3 = 4x(2x + 1)$
d $(2x + 1)(3x - 2) = 2x^2 - 3x - 9$

5 The diagram shows two rectangles.
The length and width of each rectangle are in centimetres.



a Tom says that the rectangles have equal perimeter.
He uses this fact to write an equation.
He uses this to show that the area of the largest rectangle is 98 cm^2.
Show Tom's method clearly.

b Sam says that the rectangles have equal area.
He uses this fact to write a quadratic equation.
He rearranges this to give an equation in the form $ax^2 + bx + c = 0$.
Show Sam's work clearly and write down the values of a, b and c.

Rearranging formulae

The method of 'doing the same to both sides' is used for solving linear equations. The same method is used for rearranging formulae.

Example 1 Rearrange $ax + b = c$ to give x in terms of a, b and c.

$$ax + b = c$$

The term with x has to be isolated on one side.

$$ax = c - b$$

$$x = \frac{c - b}{a}$$

Do this by subtracting b from both sides. Then, divide both sides by a.

Example 2 Solve for x: $sx + t = lx + k$.

$$sx + t = lx + k$$

$$sx - lx = k - t$$

$$x(s - l) = k - t$$

x terms are on both sides. Collect all x terms on one side, the rest of the terms on the other. Do this by subtracting lx and t from both sides. When x appears in more than one term put it outside the bracket.

$$x = \frac{k - t}{s - l}$$

Divide both sides by $(s - l)$.

Example 3 Give x in terms of a, b and c, where $\frac{x}{a} + \frac{x-1}{b} = c$.

$$\frac{x}{a} + \frac{x-1}{b} = c$$

Multiply every term by ab, the LCM of the denominators, to eliminate the fractions.

$$\cancel{a}b \times \frac{x}{\cancel{a}} + a\cancel{b} \times \frac{x-1}{\cancel{b}} = ab \times c$$

$$bx + a(x - 1) = abc$$

Remove the bracket.

$$bx + ax - a = abc$$

$$x(b + a) = abc + a$$

Collect all x terms on one side, the rest of the terms on the other.

$$x = \frac{a(bc + 1)}{a + b}$$

x appears in more than one term; put it outside the bracket, and divide by $(a + b)$.

Example 4

Solve for x: $\frac{l}{x} + m = \frac{n}{x}$.

$$\frac{l}{x} + m = \frac{n}{x}$$

Eliminate the fractions by multiplying all terms by x.

$$l + mx = n$$

Isolate the x term on one side.

$$mx = n - l$$

$$x = \frac{n - l}{m}$$

Example 5

Solve for x: $\frac{x - s}{c - d} = \frac{t - x}{y}$.

$$\frac{x - s}{c - d} = \frac{t - x}{y}$$

Two equal fractions, so cross-multiply.

$$y(x - s) = (t - x)(c - d)$$

Remove brackets.

$$xy - sy = ct - dt - cx + dx$$

Collect all x terms on one side, the rest of the terms on the other; take x outside the bracket.

$$xy + cx - dx = ct - dt + sy$$

$$x(y + c - d) = ct - dt + sy$$

Divide by $(y + c - d)$.

$$x = \frac{ct - dt + sy}{y + c - d}$$

Example 6

Rearrange $T = 2\pi\sqrt{\frac{l}{g}}$ to make g the subject of the formula.

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Square both sides to remove the root sign. *Note*: $(2\pi)^2 = 4\pi^2$

$$T^2 = 4\pi^2\frac{l}{g}$$

Multiply by g to eliminate the fraction.

$$T^2 g = 4\pi^2 l$$

$$g = \frac{4\pi^2 l}{T^2}$$

Note In this example both sides were *squared*. If the *square root* of both sides is taken remember that the root can be positive or negative. $a^2 = b \Rightarrow a = \pm\sqrt{b}$.

Example 7

This example illustrates the fact that there may be several possible forms of the answer when rearranging formulae. When checking an answer, you may have the correct answer but expressed in a different form.

Solve for t: $2(t+x)=y$.

$$2(t+x)=y$$
$$t+x=\frac{y}{2}$$
$$t=\frac{y}{2}-x$$

One method is to divide by 2, and then subtract x from both sides.

Alternatively

$$2(t+x)=y$$
$$2t+2x=y$$
$$2t=y-2x$$
$$t=\frac{y-2x}{2}$$

A second method is to remove the brackets, subtract $2x$ from both sides and then divide by 2.

The two answers are equivalent.

Exercise

1 Solve these equations for x.

a $ax-b=c$ **b** $a-x=b$ **c** $cx=d$ **d** $\frac{e}{x}=f$

e $g-hx=j$ **f** $\frac{k}{x}+l=m$ **g** $\frac{3}{x}+\frac{a}{x}=b$ **h** $\frac{x+n}{a}=b$

i $\frac{2x}{3}+b=d$ **j** $hx+jx=k$ **k** $mx+n=px$ **l** $\frac{1}{2}sx=t$

m $a(x+b)=c$ **n** $p(x+q)=rx$ **o** $3(ax+b)=2(cx+d)$

p $\frac{a}{b-x}=c$ **q** $\frac{t}{x}-p=\frac{s}{x}$ **r** $a(px+q)=c(r-sx)$

s $a=\frac{2b+3x}{3b-2x}$ **t** $a=\frac{x-b}{x+b}$ **u** $2x+2y+2mx-4my+1=0$

2 Solve these equations for x.

a $\sqrt{x}=a$ **b** $\sqrt{3x}=a$ **c** $3\sqrt{x}=a$

d $\sqrt{x}+a=b$ **e** $\sqrt{x+a}=b$ **f** $x^2=c$

g $x^2+c=d$ **h** $a\sqrt{x-b}=c$ **i** $a\sqrt{x}-1=b$

j $x=\frac{k}{x}$ **k** $\frac{d}{\sqrt{x}}=e$ **l** $\sqrt{\frac{x}{a}}=b$

m $\sqrt{x^2-c^2}+c=x$ **n** $x^{\frac{1}{3}}=y$ **o** $x^{\frac{2}{3}}=y$

p $x^{m/n}=p$ **q** $x^{-2}=q$ **r** $x^{-\frac{1}{4}}=s$

3 Transform these formulae to make the letter in the brackets the subject. Where there is more than one letter in the brackets, make each letter the subject in turn.

a $C = 2\pi r$ (r)

b $pv = c$ (p, v)

c $v = u + at$ (u, a, t)

d $A = lb$ (b)

e $\frac{pv}{t} = k$ (t, p)

f $\frac{p_1v_1}{t_1} = \frac{p_2v_2}{t_2}$ (v_2, t_1)

g $I = \frac{PRT}{100}$ (P)

h $A = \pi r^2$ (r)

i $S = \frac{1}{2}gt^2$ (t)

j $S = 4\pi r^2$ (r)

k $d = \sqrt{y}$ (y)

l $d = z^2$ (z)

m $a = \sqrt{\frac{4h}{3}}$ (h)

n $t = 2\pi\sqrt{y}$ (y)

o $t = 2\pi\sqrt{\frac{l}{g}}$ (l, g)

p $a^2 + b^2 = c^2$ (a)

q $V = \frac{1}{3}\pi r^2 h$ (h, r)

r $A = \pi(R^2 - r^2)$ (R, r)

s $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$ (u, v, f)

t $A = P + \frac{PRT}{100}$ (P, T)

u $s = \frac{n}{2}(a + l)$ (n, a)

v $A = \pi r\sqrt{h^2 + r^2}$ (h)

w $\frac{L}{E} = \frac{2a}{R - r}$ (R)

x $k = \frac{brt}{a - b}$ (b)

y $V = \frac{1}{3}\sqrt{\frac{s^3}{8\pi}}$ (s)

z $A = \frac{1}{2}m(v^2 - u^2)$ (u)

Example (Extension)

These examples solve equations involving fractions.

Solve $\dfrac{7}{x+5} - \dfrac{1}{x-3} = \dfrac{5}{3}$.

$$\frac{7}{x+5} - \frac{1}{x-3} = \frac{5}{3}$$

Multiply through by the LCM of the denominators: $3(x+5)(x-3)$.

$$7 \times 3(x-3) - 3(x+5) = 5(x+5)(x-3)$$

$$21x - 63 - 3x - 15 = 5(x^2 + 2x - 15)$$

$$18x - 78 = 5x^2 + 10x - 75$$

Arrange in the form $ax^2+bx+c=0$, $a>0$.

$$5x^2 - 8x + 3 = 0$$

$$(5x-3)(x-1) = 0$$

$$\Rightarrow \quad 5x - 3 = 0 \text{ or } x - 1 = 0$$

$$\therefore \quad x = \tfrac{3}{5} \qquad x = 1$$

So the solution is $x = \frac{3}{5}$ or $x = 1$.

Example (Extension)

Find the values of x for which $\dfrac{2x+1}{x-4} = \dfrac{2(2x+1)}{x-2}$.

$$\frac{2x+1}{x-4} = \frac{2(2x+1)}{x-2}$$

Two equal fractions so use cross-multiplying.

$$(2x+1)(x-2) = 2(2x+1)(x-4)$$

$$2x^2 - 3x - 2 = 2(2x^2 - 7x - 4)$$

$$= 4x^2 - 14x - 8$$

Arrange in the form $ax^2+bx+c=0$, $a>0$.

$$2x^2 - 11x - 6 = 0$$

$$(2x+1)(x-6) = 0$$

$$\Rightarrow \quad x = -\tfrac{1}{2} \text{ or } x = 6$$

So the values of x which satisfy the equation are $x = -\frac{1}{2}$ and $x = 6$.

Extension

Factorising a quadratic expression using the discriminant

The discriminant, $b^2 - 4ac$, can be used to decide whether $ax^2 + bx + c$ can be factorised.

$b^2 - 4ac$	$ax^2 + bx + c$
> 0 and a perfect square	can be factorised with integers
> 0 and not a perfect square	can be factorised with surds
$= 0$	can be factorised as a perfect square
< 0	cannot be factorised with real numbers

Note

If the solution of a quadratic equation is $x = \frac{2}{5}$ and $x = -\frac{4}{7}$, then, working backwards, the equation must be $(5x - 2)(7x + 4) = 0$.

Example

Factorise $2x^2 - 3x - 20$.

The discriminant of $2x^2 - 3x - 20 = 0$ is $(-3)^2 - 4 \times 2 \times (-20) = 169 = 13^2$.

Since the discriminant is a perfect square, the expression can be factorised using integers. Solving $2x^2 - 3x - 20 = 0$ gives

$$x = \frac{3 \pm \sqrt{169}}{4}$$
$$= \frac{3 \pm 13}{4}$$

$\therefore \quad x = 4$ or $x = -\frac{5}{2}$

$x = 4$ or $x = -\frac{5}{2} \Leftrightarrow (x - 4)(2x + 5) = 0.$

So (working backwards): $2x^2 - 3x - 20 = (x - 4)(2x + 5)$

Extension Exercise

1 Solve these equations.

a $9x = \dfrac{4}{x+1}$

b $\dfrac{2x-1}{1-x} = \dfrac{x-2}{x+3}$

c $\dfrac{4}{x+1} - \dfrac{2}{x-2} = -1$

d $\dfrac{x+10}{x-5} - \dfrac{10}{x} = \dfrac{11}{6}$

e $\dfrac{15}{x+1} - \dfrac{8}{x-2} = \dfrac{3}{x-5}$

f $\dfrac{1}{x-1} = \dfrac{x+1}{2x+7}$

g $x^3 - x^2 - 20x = 0$

h $x^4 - 81 = 0$

i $9x^4 + 5x^2 - 4 = 0$

j $x^4 - 1 = 0$

2 Solve these equations for x.

a $(x-a)^3 - b^2(x-a) = 0$

b $x^3 + a^2x = 0$

c $x^4 - a^4 = 0$

d $(x-p)^3 = q^3$

e $x^5 + k^2x = 0$

f $x^6 - 7k^3x^3 - 8k^6 = 0$

3 Transform these formulae to make the letter in the brackets the subject.

a $\dfrac{y-k}{K-k} = \dfrac{x-h}{H-h}$ (y)

b $2x - 3y - 3mx + 2my - 2m + 4 = 0$ (m)

c $\dfrac{c}{t} = \dfrac{x-ct}{t^2}$ (x)

d $3mc = (4+3m)(c-4)$ (c)

e $x - \dfrac{c}{t} = -t^2(x+ct)$ (x)

f $\dfrac{R-r}{R+r} = \dfrac{r+a}{r+b}$ (R)

4 Solve $(b+1)x^2 - (b^2+b+1)x + b = 0$ for x.

5 **a** Show that a necessary and sufficient condition for α and β to be the roots of a quadratic equation is $x^2 - (\alpha+\beta)x + \alpha\beta = 0$.

b By comparing $x^2 - (\alpha+\beta)x + \alpha\beta = 0$ with $ax^2 + bx + c = 0$ show that

i the sum of the roots of $ax^2 + bx + c = 0$ is $-\dfrac{b}{a}$

ii the product of the roots of $ax^2 + bx + c = 0$ is $\dfrac{c}{a}$.

c Show that if α, β and γ are the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$ then

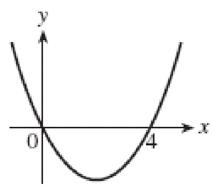
i $\alpha + \beta + \gamma = -\dfrac{b}{a}$

ii $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a}$

iii $\alpha\beta\gamma = -\dfrac{d}{a}$.

6 The graphs of five quadratic functions are shown. Find an equation for each graph.

a



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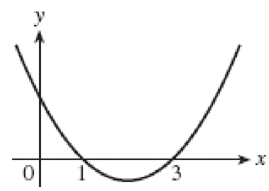

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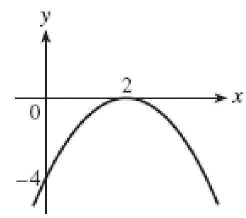

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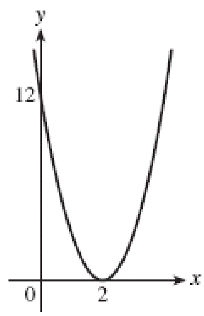

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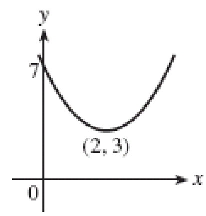


3 Equations and quadratic functions

Test yourself

1 If $2x + 1 = 3(x + 3) - 2(7 - x)$ then x is equal to

A $\frac{2}{3}$ **B** 2 **C** 3 **D** $3\frac{1}{3}$ **E** -2

2 The vertex of the parabola $y = x^2 + 6x + 2$ is at

A $(6, 2)$ **B** $(3, 2)$ **C** $(3, -7)$ **D** $(-3, -4)$ **E** $(-3, -7)$

3 If $3k^2 - 5k = 2$ then k is equal to

A 0 or $1\frac{2}{3}$ **B** -2 or $\frac{1}{3}$ **C** -3 or 2 **D** $-\frac{1}{3}$ or 2 **E** $\frac{2}{3}$ or 1

4 The roots of $2x^2 - 4x - 3 = 0$ are

A $2 \pm \sqrt{2}$ **B** $1 \pm \dfrac{\sqrt{10}}{2}$ **C** 1 and $1\frac{1}{2}$ **D** $1 \pm \sqrt{5}$ **E** $-\frac{1}{2}$ and 3

5 Given that $2x^2 + kx + 8 = 0$ has equal roots and $k > 0$, $k =$

A 0 **B** $2\sqrt{2}$ **C** 64 **D** 8 **E** 4

Inequalities

1 Put the symbols $<$ or $>$ between these pairs of integers to make a correct statement.

a $3 \quad 2$ **b** $-3 \quad 2$
c $5 \quad -1$ **d** $-2 \quad -7$
e $0 \quad -1$ **f** $-3 \quad 0$

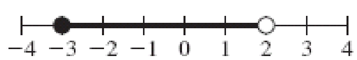
2 The inequality $-4 \leqslant x < 3$ can be shown on a number line like this.

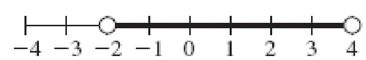
The full circle shows that 4 is included in the inequality.
The hollow circle shows that 3 is **not** included in the inequality.

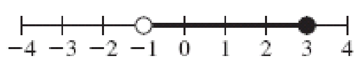
a Show each of these inequalities on a number line.

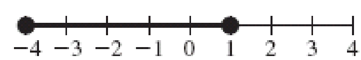
i $x > 1$ **ii** $x < 2$
iii $x \leqslant 3$ **iv** $x \geqslant -3$
v $-1 < x < 2$ **vi** $-2 \leqslant x < 4$
vii $-3 \leqslant x \leqslant 4$ **viii** $0 < x \leqslant 3$

b Write down the inequality describing the numbers shown on each of these number lines.

i 

ii 

iii 

iv 

3 List all the integers that satisfy the following inequalities.

a $-3 \leqslant x \leqslant 3$ **b** $-3 \leqslant x < 3$
c $-4 < x < 5$ **d** $-4 < x \leqslant -1$
e $-6 < x < 3$ **f** $-6 < x \leqslant -1$

4 Solve each of the following inequalities.

a $4x + 2 \leqslant 14$ **b** $5x - 3 \geqslant 17$
c $6x + 1 < 19$ **d** $5x - 3 \geqslant -18$
e $2(2x - 3) < 12$ **f** $3 + 3x \geqslant -12$
g $5 + 6x \leqslant -7$ **h** $4(2 + x) \geqslant -12$

5 **a** Draw axes with x and y taking values from -5 to 5.
On your graph shade the region satisfied by the inequality $y \leqslant 3$.

b Repeat **a** for the following inequalities:

i $y \geqslant x + 2$ **ii** $y \leqslant x + 2$
iii $x + y \leqslant 2$ **iv** $x + y \geqslant 3$

Extension Exercise

1 Solve

a $x^2 + 7x > 6 + 2x^3$ **b** $x^2(x - 3) < 1 - 3x$

2 Certain operations applied to both sides of an inequality require the inequality sign to be reversed for the statement to remain true.

Explain why the sign has to be reversed when multiplying by a negative number.

Explain also why taking reciprocals of and squaring both sides is unsafe.

3 Equations and inequalities fall into one of three categories: they are either true for *all* values of x, true for *some* values of x or *not true for any* value of x. Decide the category for each of these.

a $x^2 + 2x + 2 = (x + 1)^2 + 1$ **b** $(2x + 1)^2 > 5$

c $9x^2 \geqslant 6x - 1$ **d** $x^2 + 2x > x^2 - 2x$

e $3x + 7 < 2(x - 4) + x + 4$ **f** $3x + 5 = 7 - 3x$

g $x^2 = x$ **h** $x(x + 1) + 7 = 5 + x + x^2$

i $(x + 1)^2 = x^2 + x$ **j** $(x + 1)^2 = x^2 + 2x + 1$

k $(x + 1)^2 = x^2 + 2x + 2$

Test yourself

1 The solution of the inequality $3(2+y)-4(2y-7) \leqslant 9$ is

A $y \leqslant 5$ **B** $y \geqslant 12\frac{1}{2}$ **C** $y \geqslant -\dfrac{31}{5}$ **D** $y \geqslant \dfrac{43}{5}$ **E** $y \geqslant 5$

2 The solution of the inequality $4(x+2)^2 \geqslant (3x+1)^2$ is

A $x \leqslant 3$ **B** $x \leqslant -1$ or $x \geqslant 3$ **C** $-1 \leqslant x \leqslant 3$

D $1-\frac{1}{5}\sqrt{95} \leqslant x \leqslant 1+\frac{1}{5}\sqrt{95}$ **E** $x \geqslant -7$

3 The range of values satisfying $-7 \leqslant 2x-5 < 7$ is

A $-1 \leqslant x < 6$ **B** $-3 \leqslant x < 2$ **C** $x \leqslant -1$ or $x > 6$

D $x \geqslant -3\frac{1}{2}$ **E** $-\frac{7}{10} \leqslant x < \frac{7}{10}$

4 The range of values satisfying $y^2-2y-8>0$ is

A $3-\sqrt{17} < y < 3+\sqrt{17}$ **B** $y < -2$

C $y > -2$ **D** $y < -2$ or $y > 4$

E $-2 < y < 4$

Simultaneous equations

1 **a** **i** How many pairs of values of x and y satisfy the equation $y = x + 2$?

ii How many pairs of values of x and y satisfy the equation $y = 3x$?

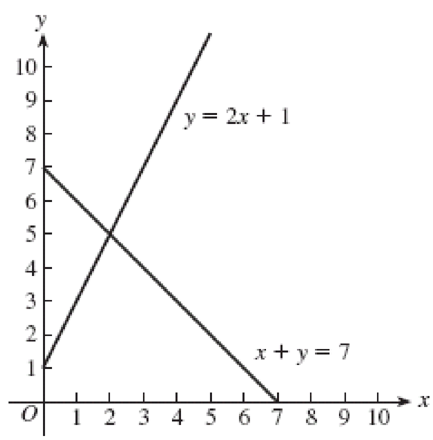
iii Find a pair of values of x and y that satisfy both the equation $y = x + 2$ and the equation $y = 3x$.

b Repeat **a** for the equations:

i $x + y = 4$ and $x + 2y = 6$.

ii $y = 3x + 4$ and $y = x^2$

2 **a** The diagram shows the graphs of $y = 2x + 1$ and $x + y = 7$



What pair of values of x and y satisfy both the equation $y = 2x + 1$ and the equation $x + y = 7$?

Hence write down the solution of the pair of simultaneous equations $y = 2x + 1$ and $x + y = 7$.

c Use a graphical method to solve the following pairs of simultaneous equations:

i $2y = 3x + 6$ and $y = 8 - x$

ii $y = 4x + 1$ and $y - 2x = 3$

3 Use the elimination method to solve each of these pairs of simultaneous equations.

a $5x - 5y = 20$
$2x + 5y = 1$

b $2x + 3y = 26$
$4x - 3y = 34$

c $2x + 6y = 14$
$x + 6y = 13$

d $2y + 3x = 9$
$4y + 3x = 12$

e $3x + 4y = 2$
$x + 2y = 0$

f $3x + 2y = 9$
$2x - 4y = -26$

4 It costs £x for an adult and £y for a child to go to the cinema.
1 adult and 4 children cost a total of £19.
3 adults and 2 children cost a total of £22.

a Form a pair of simultaneous equations to show this information.
b Solve the simultaneous equations to find the cost for an adult and a child.

5 There are x seats in a 1st class carriage and y seats in a standard class carriage.
Train P has three 1st class carriages and four standard class carriages and a total of 515 seats.
Train Q has two 1st class carriages and six standard class carriages and a total of 610 seats.

a Form a pair of simultaneous equations to show this information.
b Solve the simultaneous equations to find the number of seats in a 1st class and standard class carriage.

Exercise

This exercise can be used to consolidate skills covered in Exercise 5B.

1 A fast train leaves Manchester for London, a journey of 330 km, at 12 noon. A slow train, travelling half as fast, leaves London for Manchester at the same time. They pass each other at 2 p.m.

Find the speed of each train.

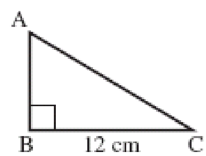
2 Siobhan is five years older than her brother. Six years ago, her brother was half her age.

Find the age of each child.

3 The momentum of a body with mass m travelling at a speed v is mv and its kinetic energy is $\frac{1}{2}mv^2$. The momentum of a certain object is $3\,\text{kg}\,\text{m}\,\text{s}^{-1}$ and its kinetic energy is 3 J.

Find the mass and the speed of the object.

4 ABC is a right-angled triangle. The base BC is 12 cm long and the hypotenuse AC is 6 cm longer than the shortest side AB.



Find the lengths of AB and AC.

5 One mobile telephone company charges £4 per month plus a pence per minute for calls; a second company charges £10 per month plus b pence per minute for calls. A customer who makes 120 minutes of calls per month has the same monthly bill for each company; a customer who makes 400 minutes of calls per month pays only half as much if he used the second company rather than the first.

Find a and b.

6 A student can complete a maths question in five minutes and a biology question in twelve minutes. A certain day's maths and biology homework consists of eleven questions altogether and takes an hour and a half.

Find how many questions were set in each subject.

7 A swimmer in a river can swim 1 km in 15 minutes when swimming with the current, but the same distance takes an hour against the current.

Find the speeds of

a the swimmer

b the current.

Extension material: Simultaneous equations

For simultaneous linear equations with more than two unknowns, the elimination method is usually used. Use one pair of equations to eliminate one variable. Then use a different pair to eliminate the same variable again. Solve the two resulting equations in the usual way.

Example 1

Solve

$$x + y + z = 7$$
$$2x + 3y - z = 0$$
$$3x + 4y + 2z = 17$$

Solution

$$x + y + z = 7 \quad (1)$$
$$2x + 3y - z = 0 \quad (2)$$
$$3x + 4y + 2z = 17 \quad (3)$$

Look for matching coefficients. The z coefficients in lines (1) and (2) match with different signs. Add lines (1) and (2) to eliminate the z terms.

(1) + (2) $\quad 3x + 4y = 7 \quad (4)$

2 × (2) $\quad 4x + 6y - 2z = 0 \quad (5)$

To obtain another equation with no z term multiply line (2) by 2 and add to line (3).

(3) + (5) $\quad 7x + 10y = 17 \quad (6)$

Now solve the equations in lines (4) and (6).

7 × (4) $\quad 21x + 28y = 49 \quad (7)$

3 × (6) $\quad 21x + 30y = 51 \quad (8)$

(8) − (7) $\quad 2y = 2$

$\therefore \quad y = 1$

Substituting in (4) $\quad 3x + 4 = 7$

$\therefore \quad x = 1$

Substituting in (1) $\quad z = 5$

Checking in (3) $\quad 3 \times 1 + 4 \times 1 + 2 \times 5 = 17$

So the solution is $x = 1$, $y = 1$, $z = 5$.

(4) (= (1) + (2)) and (1) have been used for substitution so (3) should be used for the check.

Intersection of two quadratic curves

Two quadratic curves meet in up to four points.

Example 2

This example illustrates a circle and hyperbola meeting in four points.

Find the points of intersection of $xy = 6$ and $x^2 + y^2 = 13$.

$$xy = 6 \quad (1)$$
$$x^2 + y^2 = 13 \quad (2)$$

From (1) $\quad x = \dfrac{6}{y}$

Rearrange (1) to give x in terms of y or y in terms of x.

Substituting in ②
$$\left(\frac{6}{y}\right)^2 + y^2 = 13$$
$$\frac{36}{y^2} + y^2 = 13$$

Multiply through by y^2.

$$36 + y^4 = 13y^2$$
$$y^4 - 13y^2 + 36 = 0$$
$$(y^2 - 4)(y^2 - 9) = 0$$

This is a quadratic in y^2. Substitute $z = y^2$ if preferred to give $z^2 - 13z + 36 = 0$.

$\Rightarrow$ $y = \pm 2$ or $y = \pm 3$

When $y = 2$, $x = 3$. When $y = -2$, $x = -3$.

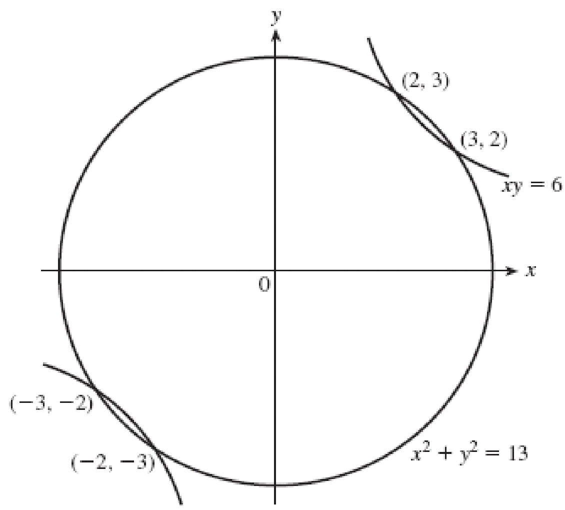
When $y = 3$, $x = 2$. When $y = -3$, $x = -2$.

Substitute in ①. Avoid substituting in equations with x^2 or y^2.

Remember to check that all solutions satisfy both equations.

The solution can be illustrated graphically.

The points of intersection are (2, 3), (−2, −3), (3, 2) and (−3, −2).



Exercise

1 A magician asks a volunteer to think of two different positive integers without telling her what they are. She then asks him to calculate x, the sum of the larger number with the square of the smaller, and y, the difference between the numbers. The volunteer tells her that $x = 9$ and $y = 3$. Find the original numbers.

2 A climbing centre has 14 fixed staff plus one additional staff member for every six climbers. There are twice as many climbers as staff altogether. Find the number of

a climbers **b** staff.

3 The temperature in a certain town fell 5°C from Monday to Tuesday. The mean temperature for the two days was 14.5°C. Find the temperature on each day.

4 In a casino, a gambler must give the banker a £20 stake at the beginning of the evening. Each time he wins a game, his amount in the bank is multiplied by a factor a; each time he loses, his amount in the bank is reduced by £b. One evening, a gambler wins his first game and loses the next ten, and finishes with £10 in the bank. A second gambler loses his first two games and wins the next one, and finishes with £30.

Find two possible pairs of values for a and b.

5 A hollow cylinder has volume $12\,\text{cm}^3$ and curved surface area $12\,\text{cm}^2$. Find

a the radius of its base **b** its height.

6 If 1 is added to the denominator of a certain fraction, the result is equal to $\frac{1}{3}$. If 1 is subtracted from the numerator of the original fraction, the result is equal to $\frac{1}{4}$. Find the fraction.

Extension Exercise

1 Solve these equations.

a $x + y - z = 8$
$4x - y + 3z = 26$
$2x + y - 4z = 8$

b $2x + y + z = 8$
$5x - 3y + 2z = 23$
$7x + y + 3z = 20$

c $x + z = 2y$
$9x + 3z = 8y$
$2x + 3y + 5z = 36$

d $\frac{1}{4}x + \frac{1}{6}y + \frac{1}{3}z = 8$
$\frac{1}{2}x - \frac{1}{9}y + \frac{1}{6}z = 5$
$\frac{1}{3}x + \frac{1}{2}y - z = 7$

2 Simultaneous linear equations may have one solution, no solution or an infinite number of solutions.

For these simultaneous linear equations, decide how many solutions there are. Illustrate each with a sketch.

a $y = 2x - 1$
$x + y = 5$

b $4x + 3y = 1$
$16x + 12y = 4$

c $x + 3y = 2$
$3x + 9y = 8$

3 Express x and y in terms of the other variables.

a $2y = x + 4c$
$5y = x + 25c$

b $ty = x + t^2$
$Ty = x + T^2$ (where $t \neq T$)

c $xy = 1$
$t^2x - y = t^3 - \dfrac{1}{t}$

d $x^2 - y^2 = 16a^2$
$y = 3x - 12a$

4 Find all possible pairs of real solutions for each of these pairs of simultaneous equations.

a $x^4 = 3y + 1$
$x^2 = y - 1$

b $x^4 - 4y = 14$
$x^2 - 2y = 3$

c $x^4 = y^2 - 19$
$x^2 = \dfrac{y^2}{10} - 1$

5 Solve these simultaneous equations.

$2(x - y) = (y + 1)^2$

$\dfrac{1}{x} = \dfrac{2}{y + 11}$

6 Solve these pairs of simultaneous equations.

a $x^2 + y^2 = 65$
$xy = 28$

b $xy = 35$
$\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{12}{35}$

c $x^2 + 4y^2 = 52$
$xy = 12$

7 The combined resistance R_T for two resistors R_1 and R_2 is given by $R_T = R_1 + R_2$ if the resistors are in series and by

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$$

if they are in parallel. A pair of resistors have a combined resistance of 9 ohms in series and 2 ohms in parallel. Find the resistance of each resistor.

8 A rectangular wooden board has sides a m and ab m.
Its area is $1.5\,\text{m}^2$ and its perimeter is 7 m. Find a and b.

9 Show that the line $y = mx$ meets the curve $x^2 + y^2 + 2gx + 2fy + c = 0$ in two real points if $(g + fm)^2 \geqslant c(1 + m^2)$.

10 Solve for x and y.

a $3x^2 - 2y^2 + 5 = 0$
$3x - 2y = 1$

b $2x^2 + 3y^2 + x = 13$
$2x + 3y = 7$

c $2x^2 - 3xy - 2y^2 = 12$
$2x - 3y = 4$

d $5x^2 - 5xy + 2y^2 = 12$
$3x - 2y + 2 = 0$

Test yourself

1 The simultaneous equations

$$4x + y = 0$$
$$2y - 3x = 11$$

have solutions

A $x = 1, y = -4$ **B** $x = \frac{11}{24}, y = \frac{11}{6}$ **C** $x = \frac{11}{14}, y = -3\frac{1}{7}$

D $x = -1, y = 4$ **E** none of these

2 A pair of simultaneous equations with one linear and one quadratic

A always has one pair of real solutions

B may have no real solutions

C always has two pairs of real solutions

D may have up to four pairs of real solutions

E always has either one or two pairs of real solutions

3 Given that $2x - 4y = 3$ and $x + 3y = 4$ then $x + y =$

A 3 **B** -1 **C** -7

D 1.8 **E** none of these

4 What are the solutions of these simultaneous equations?

$$x + 3y = 3$$
$$2y^2 + 3x + 1 = 0$$

A $x = -3, y = 2$ **B** $x = -3, y = 2$ and $x = -\frac{9}{2}, y = \frac{5}{2}$

C $x = 3, y = 0$ and $x = \frac{9}{2}, y = -\frac{1}{2}$ **D** $x = -3, y = 2$ and $x = \frac{9}{2}, y = -\frac{5}{2}$

E no real solution

5 If the straight line $y = k - 5x$ meets the curve $xy = 3$ in two distinct points then the value of k could be

A 0 **B** -3 **C** 5 **D** 7 **E** -8

6 The angles in a scalene triangle, in ascending order, are θ, 2θ and ϕ. The largest angle is 5° more than the next one. θ and ϕ are

A 30° and 70° **B** $58\frac{1}{3}^\circ$ and $121\frac{2}{3}^\circ$ **C** 35° and 75°

D 45° and 90° **E** $22\frac{1}{2}^\circ$ and 45°

7 If the line $y = 3x - k$ is a tangent to the curve $y = 2x^2 - 5$ then $k =$

A $\frac{29}{4}$ **B** $-\frac{1}{2}$ **C** $\frac{49}{8}$ **D** $-\frac{31}{8}$ **E** 0

8 The area of a rectangular room measuring a m by b m is $8\,\text{m}^2$ more than the area of a square room of side a m. The perimeter of the rectangular room is 1.25 times that of the square room.

a and b are

A 2 and 3 **B** 2 and 6 **C** 3 and $4\frac{1}{2}$ **D** 4 and 6 **E** 8 and 9

Coordinate geometry and the straight line

1 **a** Draw axes using $-10 \leqslant x \leqslant 10$ and $-10 \leqslant y \leqslant 10$.

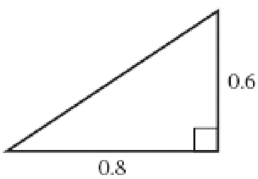
b Plot and label points A to G with coordinates A $(-5, 2)$, B $(-8, -8)$, C $(-1, 10)$, D $(-10, 10)$, E $(0, 9)$, F $(5, -2)$, G $(10, -8)$.

c Find the coordinates of the mid-points of the line segments:

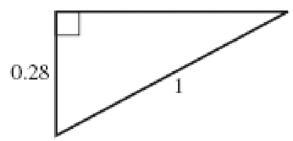
i AC **ii** BD **iii** EG **iv** CF **v** AG

2 Use Pythagoras' rule to find the unknown lengths in these triangles.

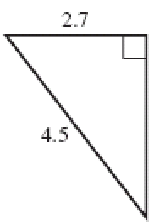
a



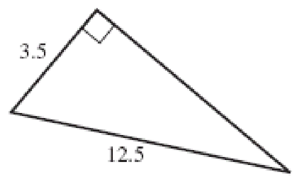
b



c



d



3 **a** Use Pythagoras' rule to find the length of the line segment joining the points with coordinates $(1, 1)$ and $(4, 5)$.

b Repeat **a** for the following pairs of points.

i $(-5, 1)$ and $(7, 6)$ **ii** $(-3, -1)$ and $(1, -4)$

iii $(6, 1)$ and $(-2, -5)$ **iv** $(-3, -12)$ and $(4, 12)$

c The line AB has length 10.
A has coordinates $(0, 6)$.
B is a point on the x-axis.
There are two possible positions for the point B.
Find their coordinates.

4 **a** Find the gradient of the line joining the point $(0, 0)$ to:

i $(1, 2)$ **ii** $(2, 1)$ **iii** $(2, 6)$ **iv** $(6, 2)$

v $(1, -2)$ **vi** $(-2, 1)$ **vii** $(2, -6)$ **viii** $(-6, 2)$

ix $(2, 3)$ **x** $(-2, 3)$ **xi** $(6, 8)$ **xii** $(6, -8)$

b Find the gradient of the lines joining:

i $(1, 2)$ to $(-3, 6)$ **ii** $(-1, -2)$ to $(3, 6)$

iii $(2, -1)$ to $(4, -9)$ **iv** $(0, -2)$ to $(4, 0)$

c **i** Draw a line with gradient 2 that passes through the point $(0, 3)$.

ii Draw a line with gradient $\frac{1}{2}$ that passes through the point $(3, 3)$.

iii Draw a line with gradient $-\frac{2}{3}$ that passes through $(1, -2)$.

5 **a** **i** Draw a line with gradient 2 that passes through (0, 0).

ii On the same diagram draw a line with gradient $-\frac{1}{2}$ that passes through (0, 0).

iii What do you notice about the two lines you have drawn?

b Repeat **a** for lines with:

i Gradients 4 and $-\frac{1}{4}$ passing though the point (0, 0).

ii Gradients -3 and $\frac{1}{3}$ passing though the point (5, 5).

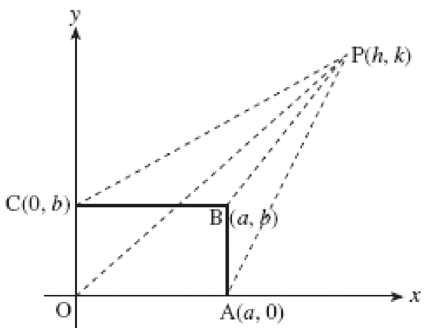
Extension material: Coordinate geometry

Many geometrical results can be proved using coordinate geometry.

Example Prove that the sum of the squares of the distances of any point, P, from two opposite vertices of a rectangle is equal to the sum of the squares of its distance from the other two vertices.

To prove the result for any rectangle and any point, a general rectangle and a general point must be chosen. It is often possible, however, to choose a simple version, e.g. to place the rectangle in a convenient position on the grid. A simple system reduces the complexity of the algebra without losing any rigour.

Proof Consider the rectangle OABC with vertices (0, 0), (a, 0), (a, b) and (0, b) respectively.



Without loss of generality, the rectangle can be placed with one vertex at the origin and sides along the axes.

Let the point, P, have coordinates (h, k).

Make P a general point by using letters for its coordinates.

To prove $PA^2 + PC^2 = PO^2 + PB^2$

State what is to be proved.

$$\begin{aligned} PA^2 + PC^2 &= (h-a)^2 + (k-0)^2 + (h-0)^2 + (k-b)^2 \\ &= h^2 - 2ah + a^2 + k^2 + h^2 + k^2 - 2bk + b^2 \\ &= 2h^2 + 2k^2 - 2ah - 2bk + a^2 + b^2 \end{aligned}$$

Write down and simplify an expression for the LHS.

$$\begin{aligned} PO^2 + PB^2 &= h^2 + k^2 + (h-a)^2 + (k-b)^2 \\ &= h^2 + k^2 + h^2 - 2ah + a^2 + k^2 - 2bk + b^2 \\ &= 2h^2 + 2k^2 - 2ah - 2bk + a^2 + b^2 \end{aligned}$$

Write down and simplify an expression for the RHS.

So LHS = RHS and the result is therefore proved.

Extension Exercise

1 Prove that the mid-point of the hypotenuse of any right-angled triangle is equidistant from its vertices.

2 Find the equations of the lines which pass through the point of intersection of the lines $x - 3y = 4$ and $3x + y = 2$, and are respectively parallel and perpendicular to the line $3x + 4y = 0$.

3 ABC is a triangle and L and M are the mid-points of the sides AB and AC respectively. Show that LM is parallel to BC and that $\text{LM} = \frac{1}{2}\text{BC}$.

4 The three straight lines $y = x$, $2y = 7x$ and $x + 4y - 60 = 0$ form a triangle. Find the equations of the three medians, and calculate the coordinates of their point of intersection.

5 The points D(2, −3), E(−1, 7) and F(3, 5) are the mid-points of the sides BC, CA, AB respectively of a triangle. Find the equations of its sides.

6 Prove that the points $(1, -1)$, $(-1, 1)$ and $(\sqrt{3}, \sqrt{3})$ are the vertices of an equilateral triangle. Find the coordinates of the point of intersection of the medians of this triangle.

7 The points A(−7, −7), B(8, −1), C(4, 9) and D are the vertices of the parallelogram ABCD. Find the coordinates of D. Prove that ABCD is a rectangle and find its area.

8 The three points O(0, 0), A(−4, 6) and B(3, 7) are the vertices of a triangle. Find the largest angle of the triangle.

9 A rhombus has three of its vertices at the points (3, 9), (2, 2) and (7, −3).

a Find the coordinates of the fourth vertex.

b Find the area of the rhombus.

10 a Given that the point P has coordinates $(h - 1, 3h + 4)$, find the coordinates P_1, P_2 and P_3 when $h = 1$, 2 and 3 respectively.

b Show that P_1, P_2 and P_3 are collinear and find the equation of the line on which the points lie.

c Show that the equation found in part **b** can also be obtained by putting $x = h - 1$ and $y = 3h + 4$ and eliminating h.

11 A, B and C are the points (1, 6), (−5, 2) and (3, 4) respectively. Find the equations of the perpendicular bisectors of AB and BC. Hence find the coordinates of the circumcentre of the triangle ABC.

12 Prove that the medians of a triangle are concurrent (i.e. meet in a point).

13 Find the circumcentre of the triangle with vertices (3, 0), (7, 8) and (3, 12).

14 Prove that the figure obtained by joining the mid-points of consecutive sides of any quadrilateral is a parallelogram.

15 Prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.

16 Find the equation of the straight line joining the feet of the perpendiculars drawn from the point (1, 1) to the lines $3x - 3y - 4 = 0$ and $3x + y - 6 = 0$.

17 Through the point A(1, 5) is drawn a line parallel to the x-axis to meet at B the line PQ whose equation is $3y = 2x - 5$. Find the length of AB and the sine of the angle between PQ and AB. Hence show that the length of the perpendicular from A to PQ is $18 \div \sqrt{13}$. Calculate the area of the triangle formed by PQ and the axes.

18 Which of these sets of points are collinear?

a $(-1, -1)$, $(1, 7)$ and $(2, 11)$

b $(0, 5)$, $(3, -1)$ and $(7, -11)$

c $(-2, 14)$, $(2, -6)$ and $(1, -1)$

Test yourself

1 The coordinates of the mid-point of (3, 4) and (a, b) are

A $(a+3, b+4)$ **B** $\left(\frac{a+b}{2}, \frac{7}{2}\right)$ **C** $(a-3, b-4)$

D $\left(\frac{a+3}{2}, \frac{b+4}{2}\right)$ **E** $\left(\frac{a-4}{2}, \frac{b-3}{2}\right)$

2 One side of a rectangle has gradient $\frac{4}{7}$. An adjacent side has gradient

A $\frac{4}{7}$ **B** $\frac{7}{4}$ **C** -1 **D** $-1\frac{3}{4}$ **E** $\frac{3}{7}$

3 The line $2x+3y+4=0$ passes through the point

A $\left(0, \frac{3}{4}\right)$ **B** $(4, 0)$ **C** $(-8, 4)$ **D** $(2, -3)$ **E** $\left(\frac{2}{3}, -3\right)$

4 The line passing through $\left(1, \sqrt{3}\right)$ and $\left(2\sqrt{3}, 6\right)$ has gradient

A $-\frac{\sqrt{3}}{6}$ **B** -6 **C** $\frac{1+2\sqrt{3}}{6}$ **D** $5-\sqrt{2}$ **E** $\sqrt{3}$

5 The line parallel to $2x+4y=3$ passing through the point $(3, -2)$ has equation

A $4x+3y=6$ **B** $11x+12y=9$ **C** $y=2x-8$

D $x+2y=-1$ **E** $2y=x-7$

6 The distance between the points $(3, p)$ and $(2p, 4)$ is

A $\sqrt{3p^2-7}$ **B** $p+1$ **C** $\sqrt{5(p^2-4p+5)}$

D $10p$ **E** $\sqrt{5p2-22p+25}$

7 The x-coordinate of the point of intersection of the lines $3y-4x+8=0$ and $2y-3x+5=0$ is

A -31 **B** 1 **C** -1 **D** $\frac{31}{17}$ **E** 31

8 The line passing through (3, 1) and (11, −3) has equation

A $y-2x+19=0$ **B** $x+2y-5=0$ **C** $y-2x+25=0$

D $x+2y+5=0$ **E** $y-2x-19=0$

9 The line $y=2-3x$ intersects the curve $x^2-y=2$ at

A $(0, 2)$ and $\left(\frac{2}{3}, 0\right)$ **B** $(1, -1)$ and $(-4, 14)$ **C** $(0, -2)$ only

D $(-3, 2)$ only **E** no points of intersection

10 The area of the triangle formed by the line $5y-6x-8=0$ and the axes is

A $\frac{32}{15}$ **B** $\frac{16}{3}$ **C** $\frac{16}{15}$ **D** $\frac{15}{64}$ **E** $\frac{32}{3}$

Graphs of functions

1 The table shows values of x and y for the equation $y = x^2 - 2x - 1$.

x	-2	-1	0	1	2	3	4
y	7	2	-1	-2	-1	2	7

Use the values in the table to draw the graph of $y = x^2 - 2x - 1$ for values of x from -2 to $+4$.

2 **a** Draw the graphs of each of these quadratic equations.

i $y = 2x - x^2$ for values of x between -2 and $+4$

ii $y = x^2 - 2x - 3$ for values of x between -3 and $+5$

iii $y = x^2 + x - 12$ for values of x between -5 and $+4$

b For each graph write down the coordinates of:

i The points where the graph intersects the x-axis.

ii The equation of the line of symmetry of the graph.

iii The minimum point on the graph.

3 For each question part, draw the graphs on the same axes and describe what happens.

a $y = x^2 + \mathrm{a}$ for the following values of a.

i 1 **ii** -1 **iii** 2 **iv** -2 **v** 3 **vi** -3

b $y = (x + \mathrm{a})^2$ for the following values of a.

i 1 **ii** -1 **iii** 2 **iv** -2 **v** 3 **vi** -3

c $y = ax^2$ for the following values of a.

i 1 **ii** -1 **iii** 2 **iv** -2 **v** 3 **vi** -3

4 **a** **i** Complete the following table of values for $y = x^3 - 7x^2 + 14x - 8$

x	0.5	1	1.5	2	2.5	3	3.5	4	4.5
y	-2.625				-1.125	-2		0	4.375

ii Use the values in the table to draw the graph of $y = x^3 - 7x^2 + 14x - 8$ for values of x from $+0.5$ to $+4.5$

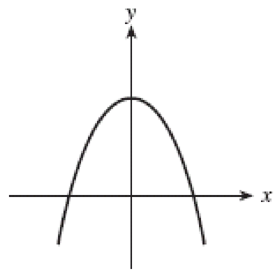
b **i** Complete the following table of values for $y = \dfrac{12}{x} + 1$

x	-6	-4	-2	-1	0	1	2	3	6
y									

ii Use the values in the table to draw the graph of $y = \dfrac{12}{x} + 1$ for values of x from -6 to $+6$.

5 Which of the following graphs is linear, quadratic, cubic or reciprocal?

a



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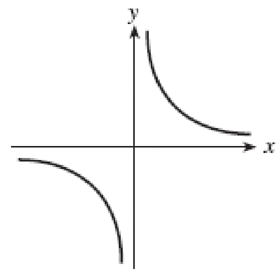

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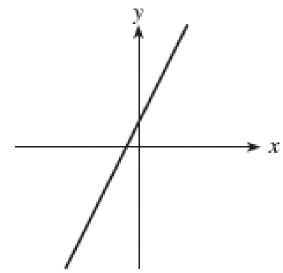

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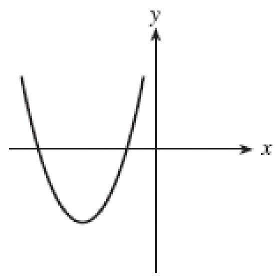

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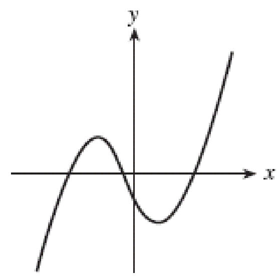
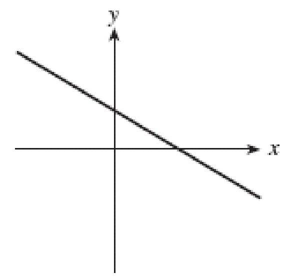


f

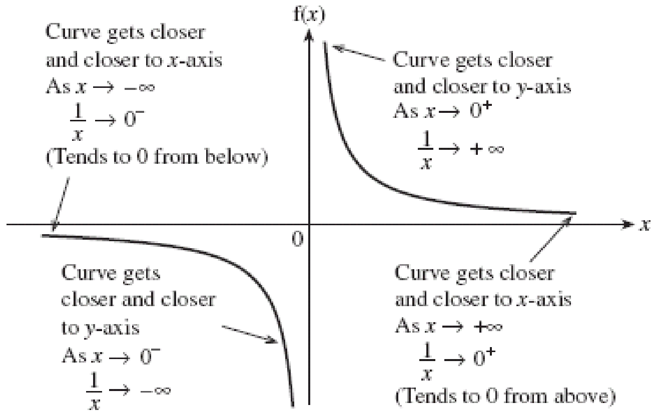


Extension material: Limits and asymptotes

In this section the idea of a limiting process, which is fundamental to the understanding of calculus, is introduced and asymptotes are discussed.

Consider the function $f(x) = \frac{1}{x}$ $\left(\text{or } y = \frac{1}{x}\right)$.

The function is not defined for $x = 0$; it is interesting, however, to investigate the function for values of x close to zero and for large values of x.



The larger the value of x, the smaller the value of $\frac{1}{x}$.

For example, when $x = 1000$, $\frac{1}{x} = \frac{1}{1000}$ and when $x = 10\,000$, $\frac{1}{x} = \frac{1}{10\,000}$.

$\frac{1}{x}$ gets closer and closer to zero the larger x gets.

$\frac{1}{x}$ tends to zero through positive values, i.e. $\frac{1}{x}$ tends to zero from above.

Expressed formally:

as $x \to \infty$, $\frac{1}{x} \to 0^+$

Read 'as x tends to infinity, $\frac{1}{x}$ tends to zero from above'.

Similarly, as $x \to 0^+$, $\frac{1}{x} \to +\infty$.

As the values of x increase, the curve approaches closer and closer to the x-axis and so the x-axis is an **asymptote** to the curve.

Note The curve gets nearer and nearer to its asymptote but never reaches it.

As the values of x approach zero from above (i.e. from the positive side of the x-axis) the curve approaches closer and closer to the y-axis. The graph shoots off to $+\infty$. As the values of x approach zero from below (i.e. from the negative side of the x-axis) the curve also approaches closer and closer to the y-axis. The graph shoots off to $-\infty$. The y-axis is also an asymptote to the curve.

Limiting value or limit

The result 'as $x \to \infty$, $\frac{1}{x} \to 0$' could be expressed as 'the limit as $x \to \infty$ of $\frac{1}{x}$ is zero'.

Expressed more formally, this is written as $\lim\limits_{x \to \infty} \frac{1}{x} = 0$

In this case, zero is the **limiting value** or **limit**.

Note Just as an asymptote is never reached, neither is a limit ever reached. The important point is to be able to get as close as you want.

Consider again the function $f(x) = \frac{1}{x}$ as $x \to \infty$ and its limiting value, zero.
Imagine this discussion:

B says to A: I want to be within $\frac{1}{1000}$ of the limit, 0. What value of x should I take?

A replies: To be within $\frac{1}{1000}$ of the limit, you could take any value of $x > 1000$.

C says to A: I want to be within $\frac{1}{10\,000}$ of the limit, 0. What value of x should I take?

A replies: To be within $\frac{1}{10\,000}$ of the limit, you could take any value of $x > 10\,000$.

If A can always answer such a question, enabling the questioner to get as close to the limit as is wanted, the function is said to tend to a limit.

Limits such as $\lim\limits_{x \to 0} \frac{1}{x}$, where the function being considered is not fully defined, are interesting to investigate.

The idea of a limit can also be applied where functions are fully defined:

e.g. $\lim\limits_{h \to 0} (2a + 2ah + h^2) = 2a$.

As h gets closer and closer to 0, $2a + 2ah + h^2$ gets closer and closer to $2a$. By choosing an appropriate value of h one can get as close to $2a$ as one wants.

The study of limits of functions is too complex to give more than an introduction at this stage. The following examples illustrate some of the possible situations, and while not comprehensive, provide tools for investigation.

Investigating limits

Limits of functions are investigated here by:

- using a calculator
- sketching a graph
- algebraic methods.

Using a calculator and/or sketching a graph can indicate what the limit is likely to be. Algebraic methods can be used to prove that the limit has a particular value.

Example 1 Investigate limits, with a calculator, or preferably using a spreadsheet.

a $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

Since the limit is required as n tends to infinity, the value of $\left(1+\frac{1}{n}\right)^n$ must be evaluated for larger and larger values of n.

The values in the table are given correct to 3 decimal places.

n	10	50	100	1000	10 000	1 000 000
$\left(1+\frac{1}{n}\right)^n$	2.594	2.692	2.705	2.717	2.718	2.718

The sequence of values for $\left(1+\frac{1}{n}\right)^n$ suggests a limit of 2.718 (correct to 3 decimal places).

This limit is one of the ways of defining the number e. This very special irrational number, given the symbol e by Euler, is useful in many area of mathematics and science. Many theories in higher mathematics, economics, statistics and probability depend on e and numerous natural phenomena in physics, chemistry and biology could not be exactly described without it.

b $\lim\limits_{x \to 0} \frac{\sin x}{x}$

Note: x is in radians, so put the calculator into radian mode.

Since the limit is required as x tends to zero, the value of $\frac{\sin x}{x}$ must be evaluated for smaller and smaller values of x.

The values in the table are given correct to 3 decimal places.

x	1	0.7	0.4	0.1	0.0001	0.000 01
$\frac{\sin x}{x}$	0.841	0.920	0.974	0.998	1.000	1.000

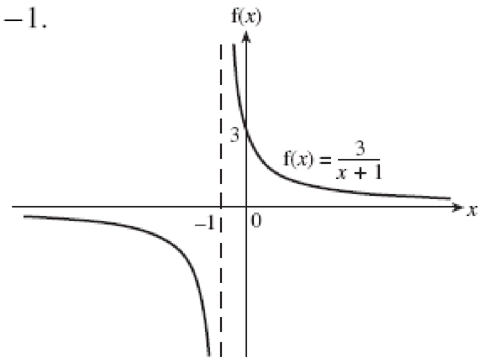
The sequence of values for $\frac{\sin x}{x}$ suggests a limit of 1.

Example 2 Investigate, by drawing a sketch, the function $f(x) = \dfrac{3}{x+1}$.

Note Graphs can be sketched using a graphics calculator, graphics package or by calculating a sufficient number of points to be able to draw a sketch or by curve sketching techniques.

The function $f(x) = \dfrac{3}{x+1}$ is not defined for $x = -1$.

The graph shows the behaviour of the function near $x = -1$, and the axes.



- As x tends to -1 from above ($x \to -1^+$), i.e. the right-hand branch of the curve, $f(x)$ tends to $+\infty$.
- As x tends to -1 from below ($x \to -1^-$), i.e. the left-hand branch of the curve, $f(x)$ tends to $-\infty$.
- As $x \to +\infty$, $f(x)$ tends to zero from above.
- As $x \to -\infty$, $f(x)$ tends to zero from below.

As $x \to \pm\infty$ the curve approaches closer and closer to the x-axis. The x-axis ($y = 0$) is an asymptote.

As $x \to -1$ the curve approaches closer and closer to the line $x = -1$. The line $x = -1$ is also an asymptote.

Note Asymptotes are not usually shown on graphics calculators. On certain calculators what appears to be asymptotes are drawn. Be wary! The calculator may be incorrectly joining up two branches of the graph.

Example 3 Investigate limits algebraically.

a $\displaystyle\lim_{x\to\infty} \frac{3x+2}{x}$

$$\frac{3x+2}{x} = 3 + \frac{2}{x}$$

Rewriting the expression makes it easy to see that the limit is 3.

As $x \to \infty$, $\dfrac{2}{x} \to 0$

$$\therefore \quad \lim_{x\to\infty}\left(\frac{3x+2}{x}\right) = 3$$

b $\lim_{x \to \infty} \frac{3x}{3+x}$

$$\frac{3x}{3+x} = \frac{3}{\frac{3}{x}+1}$$

Rewriting the expression by dividing the numerator and denominator by x enables the limit to be found.

As $x \to \infty$, $\frac{3}{x} \to 0$

$$\therefore \quad \lim_{x \to \infty}\left(\frac{3x}{3+x}\right) = 3$$

c $\lim_{a \to 0}(3x^2 + 3ax + a^2) = 3x^2$

As a gets closer and closer to 0, the terms $3ax$ and a^2 get closer and closer to 0.

d $\lim_{x \to 2} \frac{x^2-4}{x-2}$

Notice that $x - 2$ is a factor of $x^2 - 4$.

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2}$$

$(x - 2)$ can be cancelled if it is not zero, i.e. $x \neq 2$.

$$= x + 2 \qquad x \neq 2$$

$\frac{x^2-4}{x-2}$ is not defined for $x = 2$.

As $x \to 2$, $x + 2 \to 4$

$$\therefore \quad \lim_{x \to 2}\left(\frac{x^2-4}{x-2}\right) = 4$$

Extension Exercise

1 Use a calculator or spreadsheet to investigate these limits. (Work in radian mode.)

a $\lim\limits_{x \to \infty} \dfrac{2x - 7}{x - 4}$

b $\lim\limits_{x \to 2} \dfrac{x^2 + 5x - 14}{x - 2}$

c $\lim\limits_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^x$

d $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$

2 Use algebraic methods to find these limits, if they exist.

a $\lim\limits_{x \to \infty} \dfrac{5x - 1}{10 + 2x}$

b $\lim\limits_{x \to \infty} \dfrac{x + 1}{x^2}$

c $\lim\limits_{x \to \infty} \dfrac{x^2 + 1}{x}$

d $\lim\limits_{x \to \infty} \dfrac{5}{1 + x}$

3 Find the limits of these expressions as $x \to 5$, if they exist.

a $\dfrac{x^2 - 4x - 5}{x - 5}$

b $\dfrac{x^2 - 25}{x - 5}$

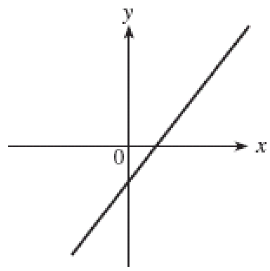
c $\dfrac{x^3 - 125}{x - 5}$

d $\dfrac{x^2 - 25}{(x - 5)^2}$

Test yourself

1 A sketch of the graph $y = mx + c$, with $m < 0$ and $c > 0$, could be

A



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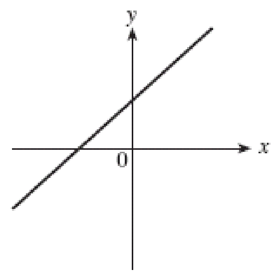

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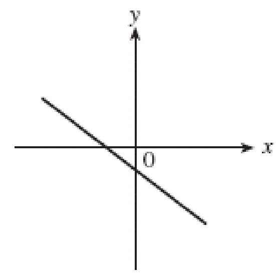

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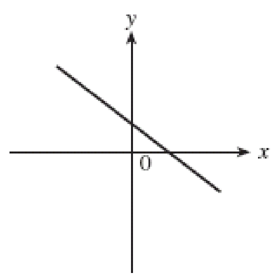

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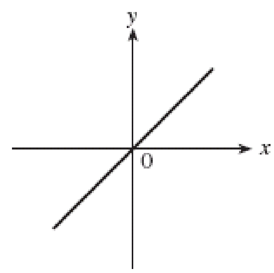
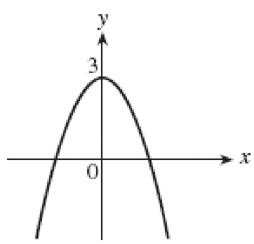


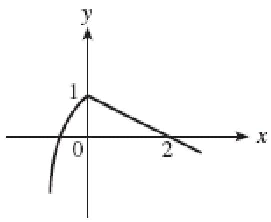
2 The equation of this graph could be



A $y = x^2 + 3$ **B** $y = x^2 - 3$ **C** $y = 3 - x^2$

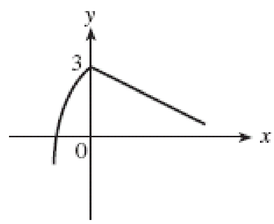
D $y = (x - 3)^2$ **E** $y = -(x^2 + 3)$

3 This is the graph of $y = g(x)$.



The graph of $y = g(x + 2)$ looks like

A



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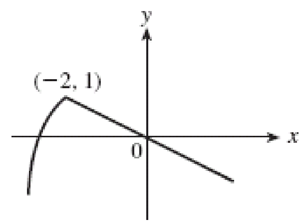

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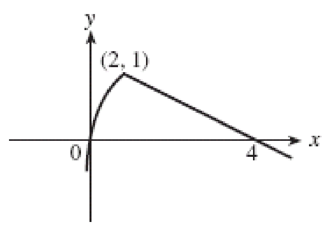

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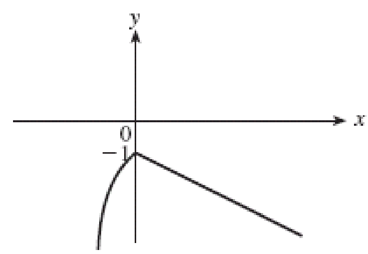

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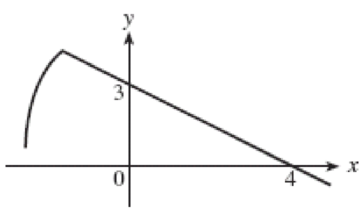
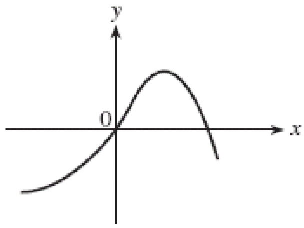
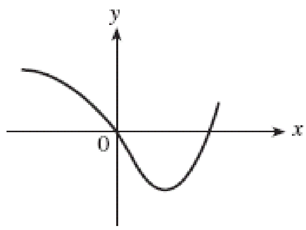


4 This is the graph of $y = f(x)$.



The graph of $y = f(-x)$ looks like

A



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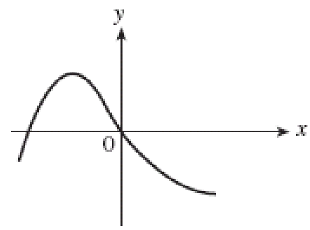

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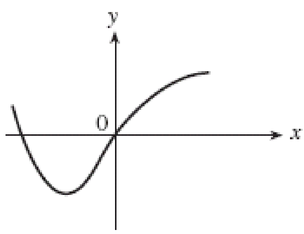

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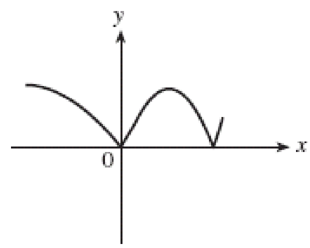

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